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On the Performance Analysis of Multihop Packet Radio Systems: Part III - Fully Connected Configurations Employing Slotted ALOHA

I. Introduction

In part II of this series [1], we analyzed the so-called star configuration in which each repeater is in line-of-sight and within range of only the station. In this note we consider the fully connected (FC) network configuration in which all repeaters are within range and in line-of-sight of each other and of the station. (See Fig. 1) With each repeater is associated a population of terminals generating traffic which is destined to the station. The main difference that exists between this and the star configuration is that in the fully connected case an arrival to a repeater in a slot will not be successfully received if any of the repeaters is actively transmitting in that slot. Each repeater is provided with a finite storage capacity which can accomodate exactly one packet. No consideration will be made here of storage capacity greater than one since the results of Part II have shown that the system is mostly channel bound and not storage bound, and since it is even more so with fully connected configurations as illustrated below. Definitions, system assumptions, and the notation used are identical to those given in Part II; they will not be repeated here. The basic performance measures sought are the system capacity and the throughput-delay tradeoff. The results will be compared to those obtained with the star configuration.

2. Transmission Protocols

Just as in Part II, the time axis is assumed to be universal and slotted into segments whose duration is equal to the transmission time of a packet. All devices are synchronized and start their packet transmission at the beginning of a slot. When its buffer is non-empty, a repeater transmits its packet in a slot with probability p. When the packet transport is successful, the packet buffer is freed; otherwise the packet remains in the repeater's buffer and thus incurs a retransmission delay geometrically distributed with mean 1/p. In fact with this protocol even the first transmission of a newly received packet (at the repeater) incurs a geometrically distributed delay following its reception with mean 1/p. We shall refer to this transmission protocol as the delayed-first-transmission (DFT) protocol. It is precisely the DFT protocol that was considered in the analysis of star-configurations in Part II.

A slight variation of the above transmission protocol is also considered here, which consists of transmitting (with probability one) a newly received packet immediately following its reception. In case of an unsuccessful transmission the packet remains in the repeater's buffer and, as above, incurs the geometrically distributed delay. This protocol will be referred to as the immediate-first-transmission (IFT) protocol. The motivation in considering it is simply an expected decrease in packet network delay due to the avoidance of an initial delay at the first transmission of the packet. Numerical results will be discussed in section 4 below.

3. Analysis of Fully Connected Configurations

3.1. DFT Protocol

With single-packet buffers, the state of the system in a slot is entirely described by the number of active repeaters. We let n^t denote this number in

slot t. n^t is a Markov chain with transition matrix P whose $(i,j)^{th}$ element is given by

$$p_{ij} \begin{cases} 0 & j<i-1 \\ P_s(i) & j=i-1 \\ (1-p)^i (1-\lambda)^{N-i} + [1-(1-p)^i - P_s(i)] & j=i \\ (1-p)^i \binom{N-i}{j-i} \lambda^{j-i} (1-\lambda)^{N-j} & j>i \end{cases} \qquad (1)$$

where $P_s(i)$ is the probability of a successful transmission, given i active repeaters, and is expressed as

$$P_s(i) = ip\,(1-p)^{i-1} \qquad (2)$$

Let $\pi_i \triangleq \lim_{t\to\infty} \Pr\{n^t=i\}$. We compute the stationary distribution $\Pi = \{\pi_0,\pi_1,\ldots,\pi_N\}$ by solving recursively the system $\Pi = \Pi p$. The average number of repeaters $\bar{n}$ is given by:

$$\bar{n} = \sum_{k=0}^{N} k\pi_k \qquad (3)$$

Let β denote the probability that a terminal transmission is blocked due to transmission by one or more repeaters. Given that k repeaters are active, this probability is simply $1 - (1-p)^k$. Removing the condition we get

$$\beta = 1 - \sum_{k=0}^{N} \pi_k (1-p)^k \qquad (4)$$

Let α denote the probability that a terminal transmission is blocked due to the repeater's buffer being full, assuming of course that no repeater is transmitting. Given k active repeaters, this probability is simply $\frac{k}{N}(1-p)^k$,

where $\binom{N-1}{k-1} \Big/ \binom{N}{k} = k/N$ is the probability that a particular repeater R_i is active. Removing the condition we get

$$\alpha = \sum_{k=0}^{N} \pi_k \frac{k}{N} (1-p)^k \tag{5}$$

The total blocking probability is given by

$$B = \alpha + \beta \tag{6}$$

The network throughput S is expressed as

$$S = \sum_{k=0}^{N} \pi_k \, kp(1-p)^{k-1} = N\lambda (1 - B) \tag{7}$$

and the network delay is simply given by

$$D_n = \frac{\bar{n}}{S} \tag{8}$$

As in part II, the access delay D_a is estimated by

$$D_a = \frac{1}{1 - B} D_{S\text{-}ALOHA}(\lambda) + \frac{B}{1 - B} \frac{Kopt}{2} \tag{9}$$

3.2 IFT Protocol

Let n^t still denote the number of active repeaters in slot t. In this protocol, n^t is not a Markov chain since its transitions depend not only on n^{t-1}, but also on whether or not new arrivals had occurred in slot t-1. Instead of formulating a Markov chain model for the system by increasing the state description to include an indicator for such events, we choose to utilize the imbedded Markov chain technique and derive the steady-state performance measures via a "cycle analysis".

Denote by empty slot a slot in which no repeater undertook a transmission. Denote by d^k the number of active repeaters in the system at the end of the k^{th} non-empty slot (see Fig. 1). d^k is a Markov chain. We derive its transition probabilities in the following.

Let $p_{ij}^k \triangleq \Pr\{d^{k+1} = j \mid d^k = i\}$. Let $P = (p_{ij})$ be the transition matrix. (We drop the superscript k as we are only interested in steady-state conditions.) For $i = 0$, we have

$$p_{0j} = \begin{cases} \dfrac{N\lambda\,(1-\lambda)^{N-1}}{1-(1-\lambda)^N} & j=0 \\ 0 & j=1 \\ \binom{N}{j} \dfrac{\lambda^j\,(1-\lambda)^{N-j}}{1-(1-\lambda)^N} & j=2,3,\ldots,N \end{cases} \qquad (10)$$

Given that $d^k = i$, let I_i denote the number of empty slot separating two consecutive non-empty slots. Note that, in a fully connected configuration, it is only in an empty slot that an arrival from a terminal can be successfully received at the repeater. Also note that with the IFT protocol, an arrival in an empty slot ends the sequence of empty slots separating two consecutive non-empty slots. Thus, for $i \neq 0, N$, we have

$$\Pr\{I_i = 0\} = 1 - (1-p)^i \qquad (11)$$

$$\Pr\{I_i > 0\} = (1-p)^i \qquad (12)$$

and the transition probabilities are given by ($i \neq 0, N$)

$$P_{ij} = \begin{cases} 0 & j < i-1 \\ \Pr\{I_i=0\} \dfrac{ip(1-p)^{i-1}}{1-(1-p)^i} + \Pr\{I_i>0\} \dfrac{ip(1-p)^{i-1}(1-\lambda)^{N-i}}{1-(1-\lambda)^{N-i}(1-p)^i} & j=i-1 \\ \Pr\{I_i=0\} \dfrac{1-(1-p)^i-ip(1-p)^{i-1}}{1-(1-p)^i} & \\ \quad + \Pr\{I_i>0\} \dfrac{(N-i)\lambda(1-\lambda)^{N-i-1}(1-p)^i+(1-\lambda)^{N-i}[1-ip(1-p)^{i-1}-(1-p)^i]}{1-(1-\lambda)^{N-i}(1-p)^i} & j=i \\ \Pr\{I_i>0\} \dfrac{(N-i)\lambda(1-\lambda)^{N-i-1}[1-(1-p)^i]}{1-(1-\lambda)^{N-i}(1-p)^i} & j=i+1 \\ \Pr\{I_i>0\} \dfrac{\binom{N-i}{j-i}\lambda^{j-i}(1-\lambda)^{N-j}}{1-(1-\lambda)^{N-i}(1-p)^i} & j \geq i+2 \end{cases} \tag{13}$$

Finally for $i = N$ we simply have

$$P_{N,j} = \begin{cases} 0 & j<N-1 \\ \dfrac{Np(1-p)^{N-1}}{1-(1-p)^N} & j=N-1 \\ 1 - \dfrac{Np(1-p)^{N-1}}{1-(1-p)^N} & j=N \end{cases} \tag{14}$$

Let $\pi_i^d \overset{\Delta}{=} \lim_{k\to\infty} \Pr\{d^k = i\}$. The stationary distribution $\pi^d = \{\pi_0^d, \pi_1^d, \ldots, \pi_N^d\}$ is obtained by solving recursively the system $\pi^d = \pi^d P$. We now derive the stationary performance measures. To do so, we define a cycle to be the interval of time separating two consecutive imbedded points. A cycle is entirely determined by the state of the system at the imbedded point which initiates it and can be labeled by that state. Given that the latter is i

the cycle length is equal to $I_i + 1$. To compute the average cycle length, we need to determine the average length of I_i which we denote by $\bar{I}_i$. The probability density function of I_i is given by

$$\Pr\{I_i=\ell\} = \begin{cases} (1-\lambda)^{N(\ell-1)}[1-(1-\lambda)^N] & i=0;\ \ell\geq 1 \\ 1-(1-p)^i & i\neq 0,N;\ \ell=0 \\ (1-p)^i[(1-\lambda)^{N-i}(1-p)^i]^{\ell-1}[1-(1-\lambda)^{N-i}(1-p)^i] & i\neq 0,N;\ \ell>0 \\ (1-p)^{N\ell}[1-(1-p)^N] & i=N;\ \ell\geq 0 \end{cases} \tag{15}$$

Thus $\bar{I}_i$ is expressed as

$$\bar{I}_i = \begin{cases} \dfrac{1}{1-(1-\lambda)^N} & i=0 \\ \dfrac{(1-p)^i}{1-(1-p)^i(1-\lambda)^{N-i}} & i\neq 0,N \\ \dfrac{(1-p)^N}{1-(1-p)^N} & i=N \end{cases} \tag{16}$$

The average throughput over the cycle, which we denoted by S_i, is precisely the probability of a successful transmission and is given by

$$S_i = \Pr\{I_i=0\}\frac{ip(1-p)^{i-1}}{1-(1-p)^i} + \Pr\{I_i>0\}\frac{(N-i)\lambda(1-\lambda)^{N-i-1}(1-p)^i + ip(1-p)^{i-1}(1-\lambda)^{N-i}}{1-(1-p)^i(1-\lambda)^{N-i}} \tag{17}$$

The average of the sum of active repeaters over the cycle is denoted by σ_i and is given by

$$\sigma_i = i\bar{I}_i + i + \Pr\{I_i>0\} \frac{(N-i)\lambda}{1-(1-p)^i(1-\lambda)^{N-i}} \tag{18}$$

By renewal theory arguments, the stationary system throughput is expressed as

$$S = \frac{\sum_{i=0}^{N} \pi_i S_i}{\sum_{i=0}^{N} \pi_i (\bar{I}_i+1)} \tag{19}$$

and the stationary average number of active repeaters is given by

$$\bar{n} = \frac{\sum_{i=0}^{N} \pi_i \sigma_i}{\sum_{i=0}^{N} \pi_i (\bar{I}_i+1)} \tag{20}$$

By Little's result, the average network delay is

$$D_n = \frac{\bar{n}}{S} \tag{21}$$

The probability of blocking B is simply

$$B = 1 - \frac{S}{N\lambda} \tag{22}$$

The access delay is again estimated by the expression given in Eq (9).

4. Numerical Results and Discussion

We start by examining the fully-connected DFT case. Contrary to what was shown for the DFT star configuration (with M=1), given λ, we note that the value of p which yields minimum D_n does not correspond to the value of p

which yields minimum blocking B (and thus maximum throughput). We get the optimum D_n for a given throughput S by plotting in the (S,D_n) plane the constant λ contours (varying p), and then by taking the lower envelope. Fortunately, the difference between the minimum blocking and the blocking achieved at optimum delay is rather insignificant! Optimum D_n and optimum B will therefore yield the optimum total delay D for a given throughput S.

In Fig 3 we plot the optimum D_n versus S for various values of N along with the corresponding curves obtained in the star configuration. Fig 4. shows the optimum blocking versus S. We note that, as expected, the probability of blocking is consistently higher for the fully connected configuration; this is simply due to the fact that transmissions by all repeaters contribute to the blocking of an incoming packet. Moreover, little discrepancy is observed as N varies between 2 and 10. The delay D_n, however, is smaller for lower throughput (with the exception of N=2), and the difference becomes more significant as N gets larger.

To see the importance of blocking due to repeaters transmissions (β) relative to the blocking due to lack of storage (α), we plot in Fig 5 α versus β for various values of N along with the curves corresponding to the star configuration; it is all too evident from this figure that β is the predominant factor and thus that the fully connected configuration is even more "channel bound" than the star configuration. Moreover, β becomes more and more important relative to α as N increases, due to the larger number of contending devices. This certainly justifies the absence of consideration in the present study for buffer sizes larger than 1.

For a two-hop fully connected environment, the system capacity is obtained for $\lambda = 1/e$. We plot in Fig 6 the DFT system capacity versus N for this and the star configuration. The fully-connected environment provides a smaller network capacity than the star configuration, especially for the smaller values of N ($N \leq 6$); however, as N gets larger ($7 \leq N \leq 10$), the capacity of the fully-connected system approaches the one achieved in the star configuration.

The total packet delay $D = D_a + D_n$ for the fully connected DFT system is plotted versus S in Figures 7, 8 and 9 for the cases N = 2, N = 5 and N= 10 respectively, along with, for comparison, the throughput delay curves corresponding to the star configuration. (The curves corresponding to the IFT protocol appearing in these figures will be discussed subsequently). We note that for both N = 2 and N = 5 the delay is larger than or equal to the delay obtained with the star network; for N = 10, however, not only does the system capacity approach the one obtained with the star configuration, but the delay is also smaller for a wide range of S; this is simply explained by the fact that, as N gets larger, the value of λ that achieves a given throughput is smaller, and thus D_n becomes the predominant component of D; the improvement in D_n observed for N = 10 (see Figure 3) overcomes the degrading effect of the larger blocking proability experienced (see Fig 4). The throughput delay curves for networks of arbitrary connectivity employing the DFT transmission protocol are expected to lie between the two displayed. However we have no formal proof for this claim!

Consider now the fully-connected IFT case*. The main focus here is to

*The comment we made earlier regarding minimum blocking and minimum delay in the fully-connected DFT case is also valid here.

compare the performance obtained with this case to the one obtained with the DFT protocol. This we do by first plotting D_n versus S in Fig 10 and B versus S in Fig 11 (at optimum) along with the delay and blocking corresponding to the DFT protocol. We note that for the most interesting range of S, D_n (and to a certain extent B) are indeed smaller with IFT. The IFT-system capacity for a two-hop environment, however, is dominated by the DFT-system capacity (with the exception of N = 2) as shown in Fig 6 above; this capacity is not too sensitive to variations in the size of the network, N. The throughput-delay curves appear in Figures 7, 8 and 9 for N = 2, N = 5 and N = 10 respectively. For N = 2, the IFT delay curve is consistently lower than the DFT curve. For N = 5 and 10, the IFT delay is lower over a significant range of the throughput, but as S increases, the relationship reverses as the IFT system reaches its capacity sooner. Thus we experience with the IFT protocol a slightly improved packet delay but a slightly degraded system capacity.

5. Conclusion

In this note we studied fully connected configurations employing slotted ALOHA. We considered two transmission protocols, the delayed-first-transmission (DFT) protocol (also considered in Part II for the analysis of star configurations) and the immediate-first-transmission (IFT) protocol. The major conclusions follow. The DFT fully connected system provides a capacity which is smaller than the one achieved by the star configuration. The difference decreases as N approaches 10. The total packet delay is also larger in the fully connected case except with the larger values of N (namely N = 10), for which a slight improvement is gained over a significant

range of S, due to a smaller achievable network delay. The sensitivity of the system performance to variations in the transmission protocol was observed by comparing the results obtained with the IFT protocol to those obtained with the DFT protocol. We basically noted a lower system capacity with IFT; this capacity is not too sensitive to changes in N. The packet delay, however, has slightly improved over a significant range of the throughput.

In the forthcoming Part IV [2], we shall be analysing the performance of packet radio networks employing the nonpersistent CSMA mode.

References

[1] Tobagi, F., "On the Performance Analysis of Multihop Packet Radio Systems: Part II - Star Configurations Employing Slotted ALOHA," Packet Radio Temporary Note #247, SRI International, Menlo Park, California, January 17, 1978.

[2] Tobagi, F., "On the Performance Analysis of Multihop Packet Radio Systems: Part IV - Fully Connected Configurations Employing CSMA," Packet Radio Temporary Note #249, SRI International, Menlo Park, California, January 31, 1978.

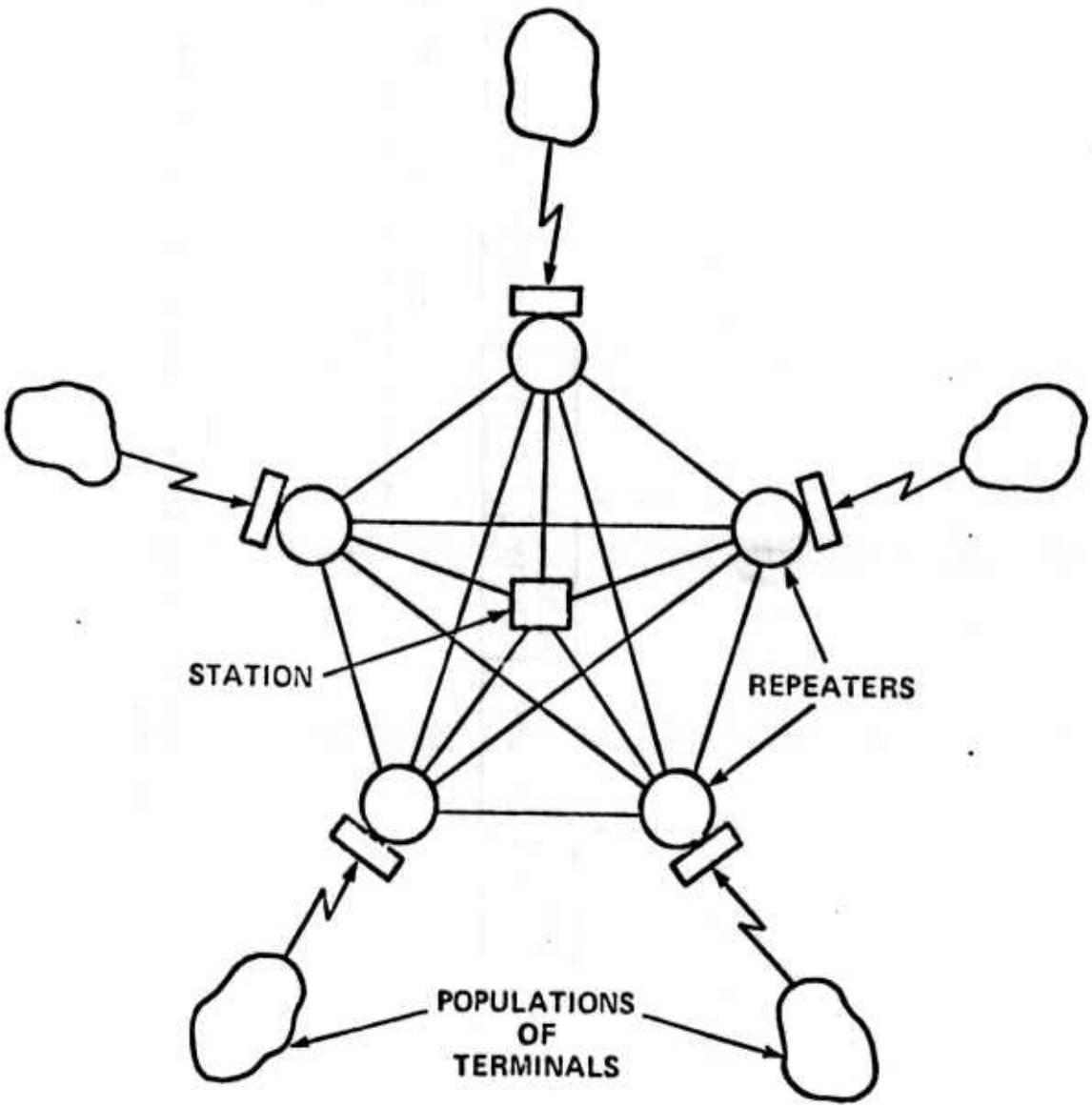

Figure 1 A Two-Hop Fully-Connected Configuration.

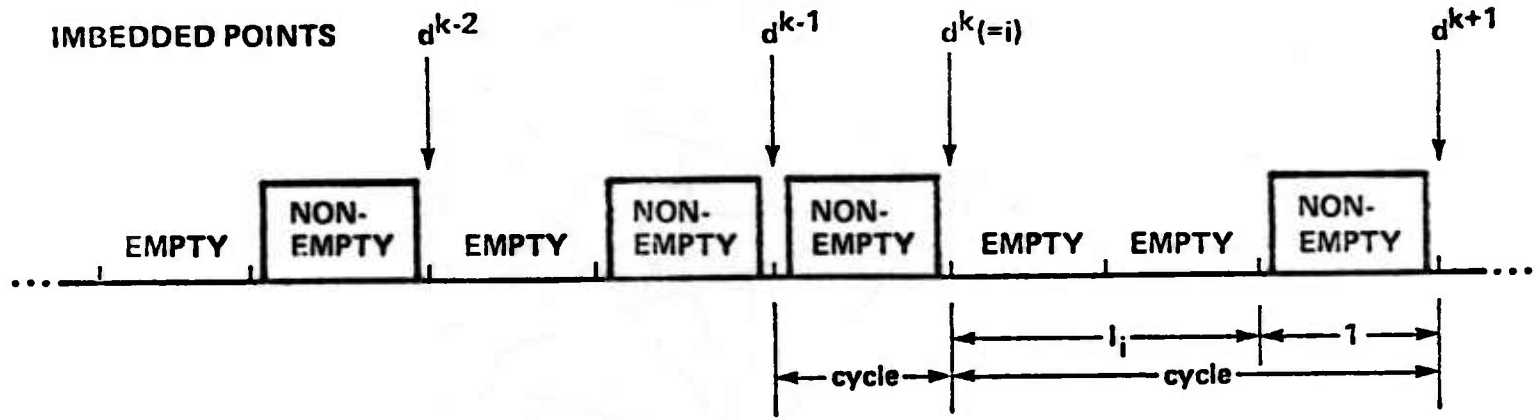

Figure 2 The Imbedded Markov Chain in the Slotted ALOHA IFT Protocol.

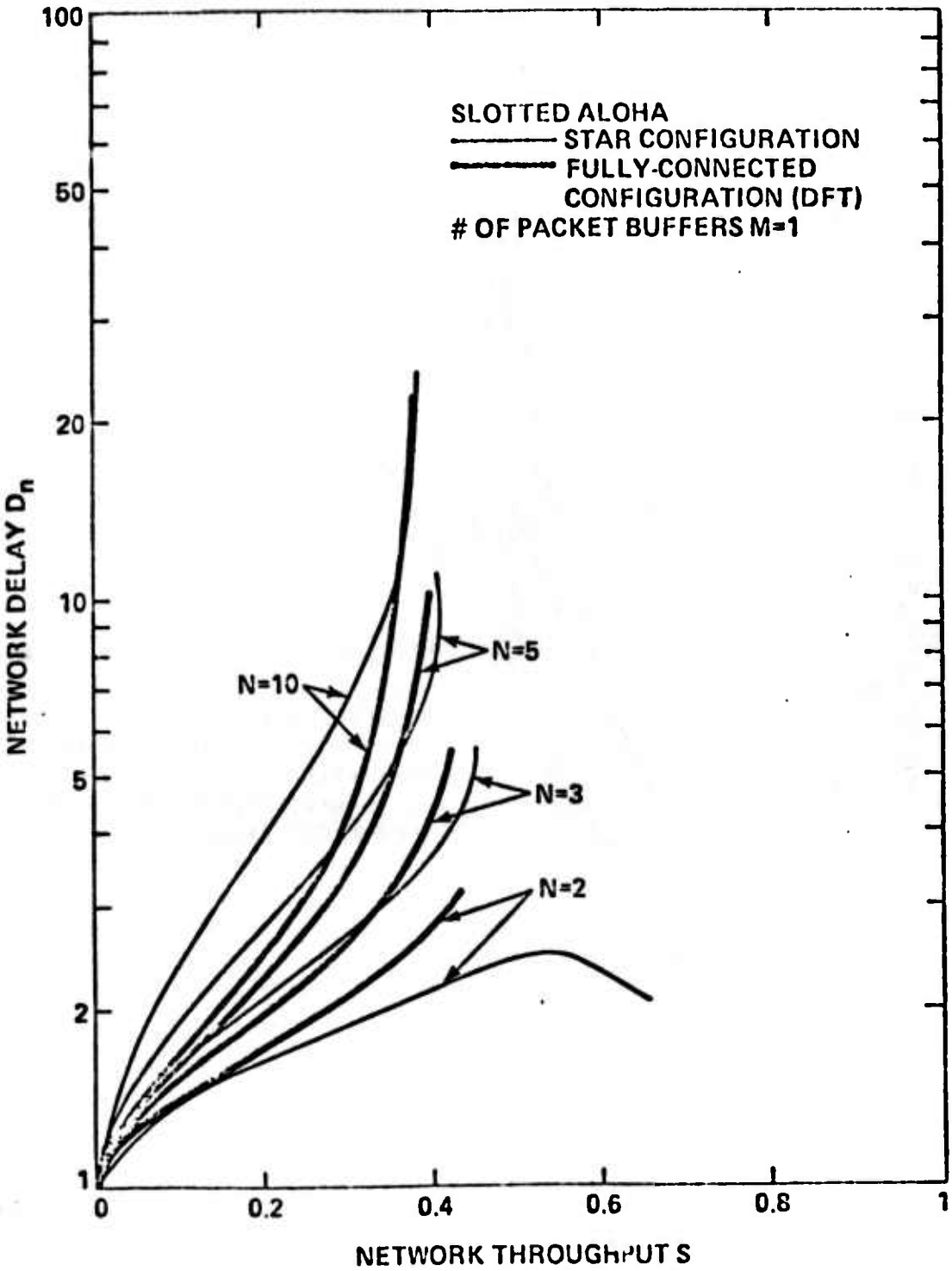

Figure 3 Slotted ALOHA Star and Fully-Connected Configurations: Optimum (Network) Throughput-Delay Curves.

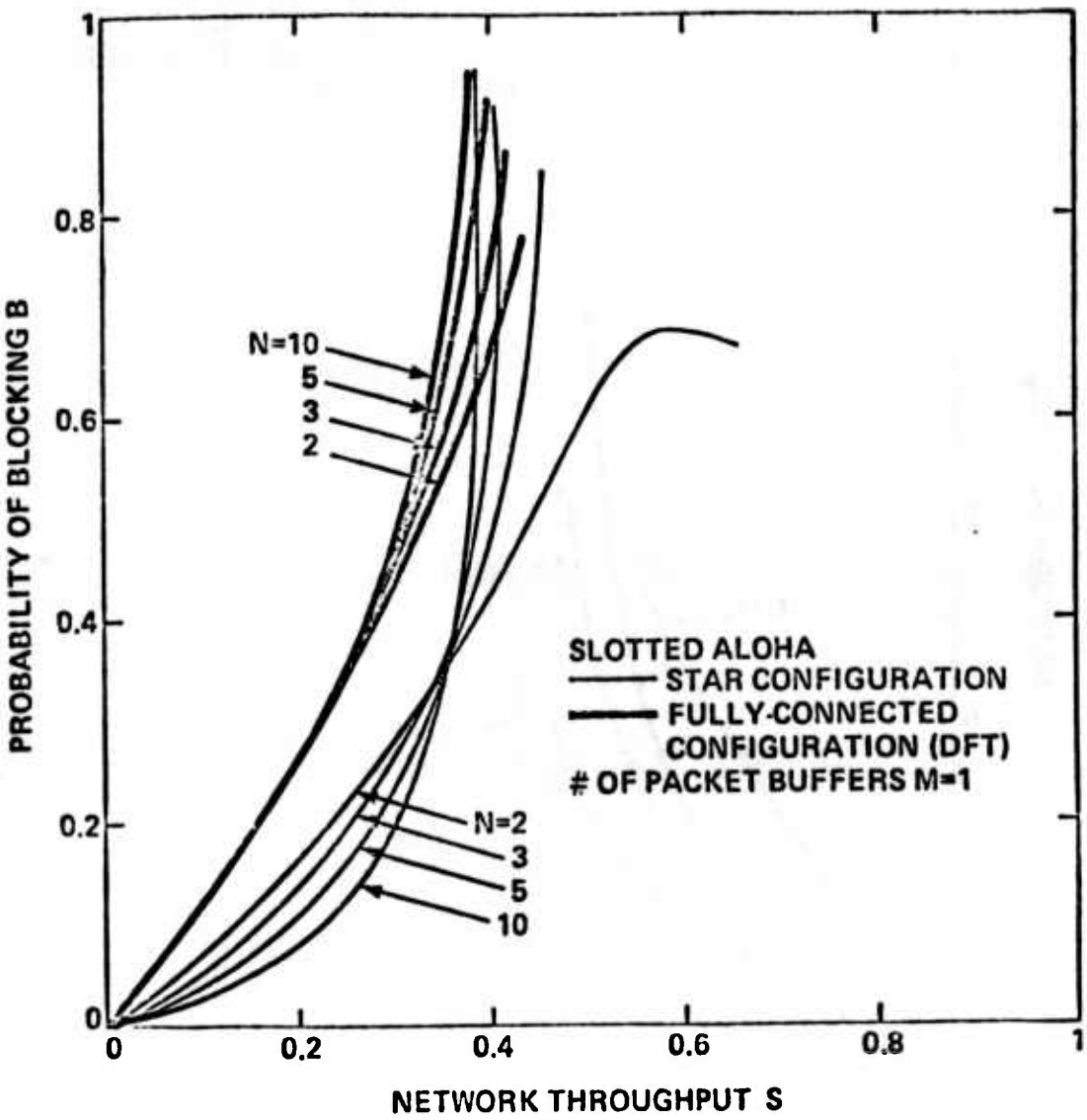

Figure 4 Slotted ALOHA Star and Fully-Connected Configurations: Minimum Blocking versus Throughput.

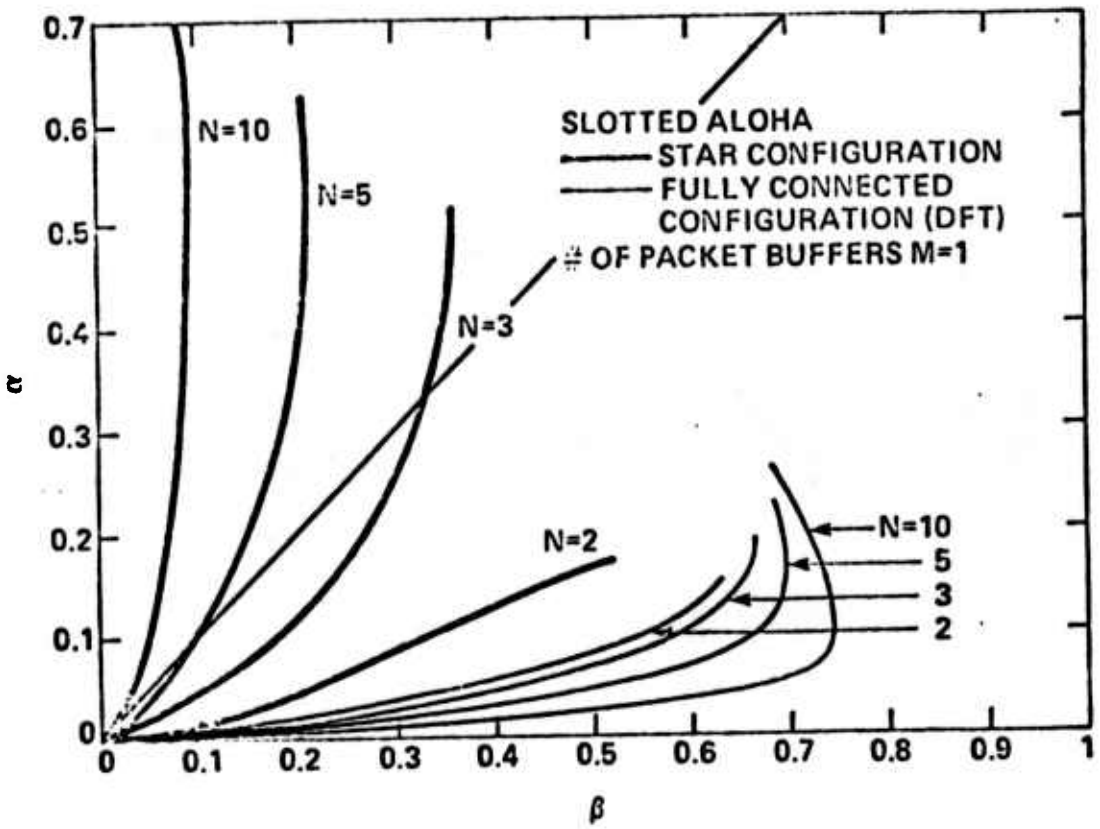

Figure 5 Slotted ALOHA Star and Fully-Connected Configurations: α versus β at Minimum Blocking.

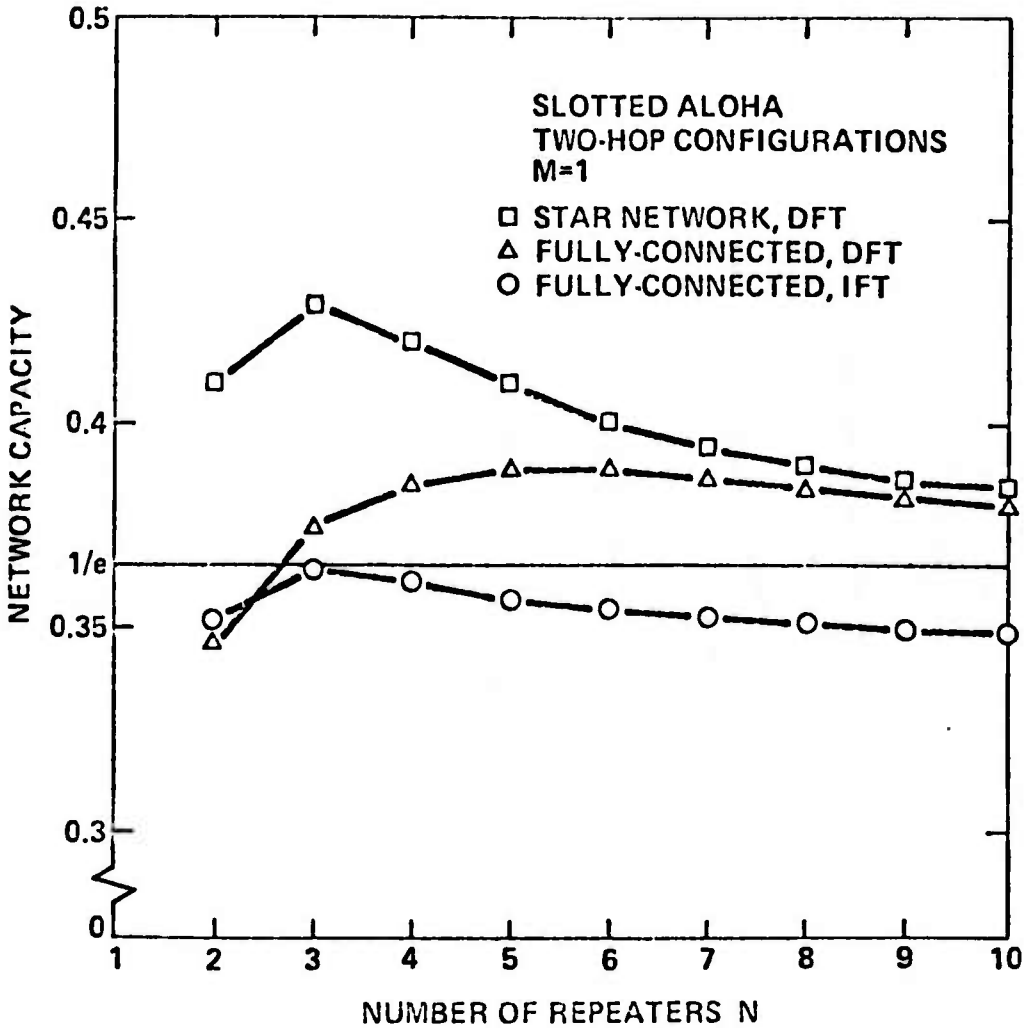

Figure 6 Slotted ALOHA Two-Hop Star and Fully-Connected Networks: Network Capacity versus N.

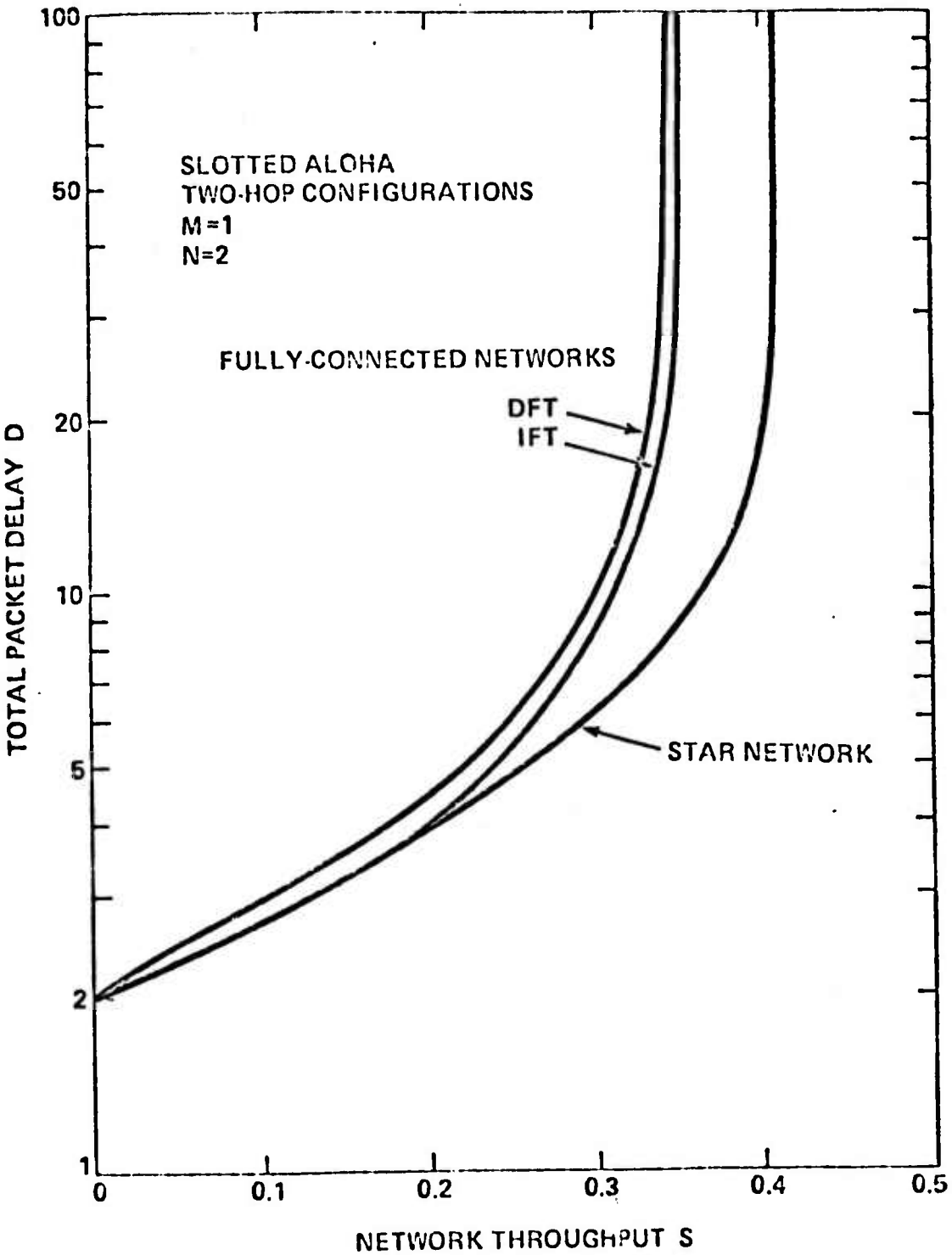

Figure 7 Throughput-Delay Tradeoff in Two-Hop Slotted ALOHA Star and Fully-Connected Networks. (N = 2)

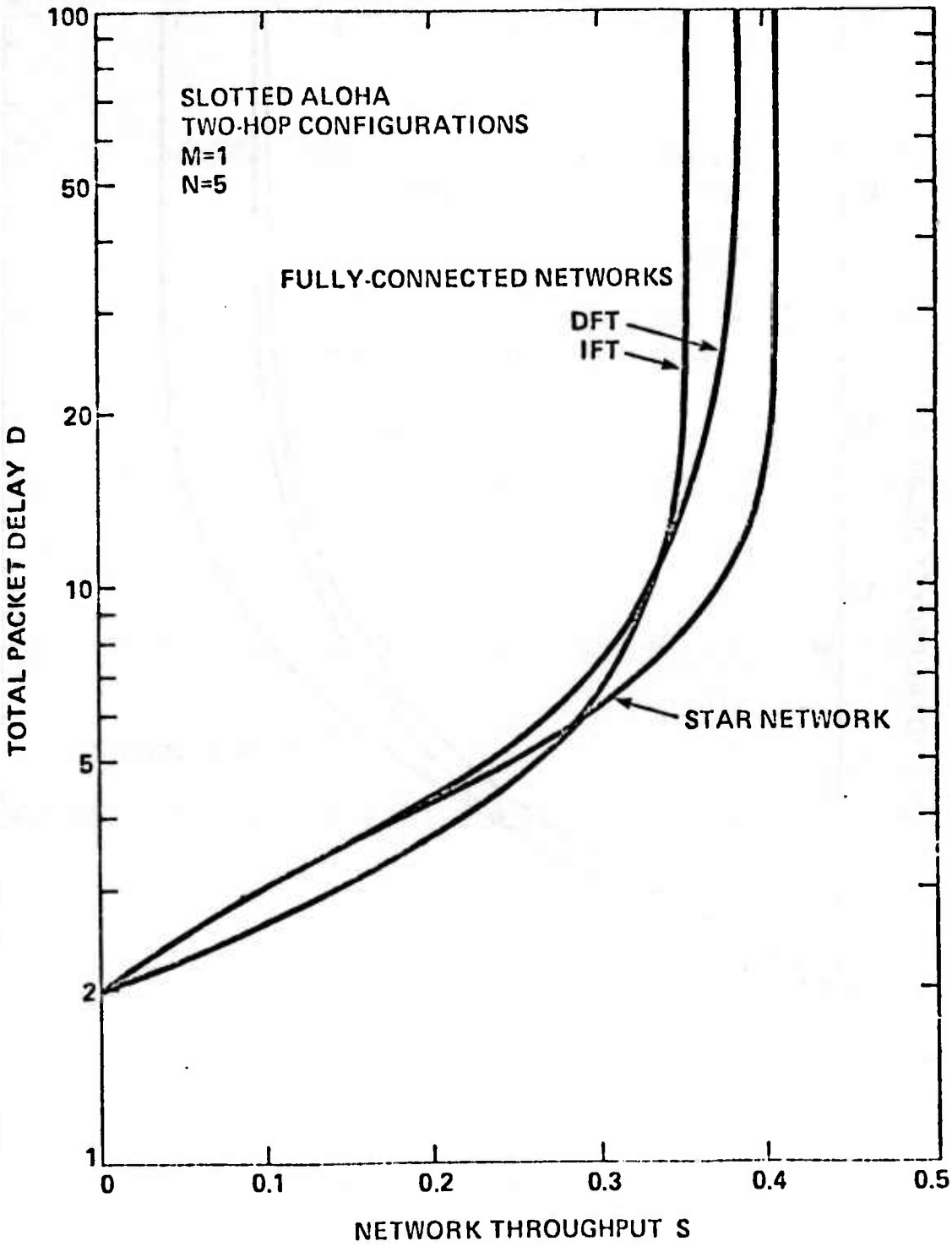

Figure 8 Throughput-Delay Tradeoff in Two-Hop Slotted ALOHA Star and Fully-Connected Networks. (N = 5)

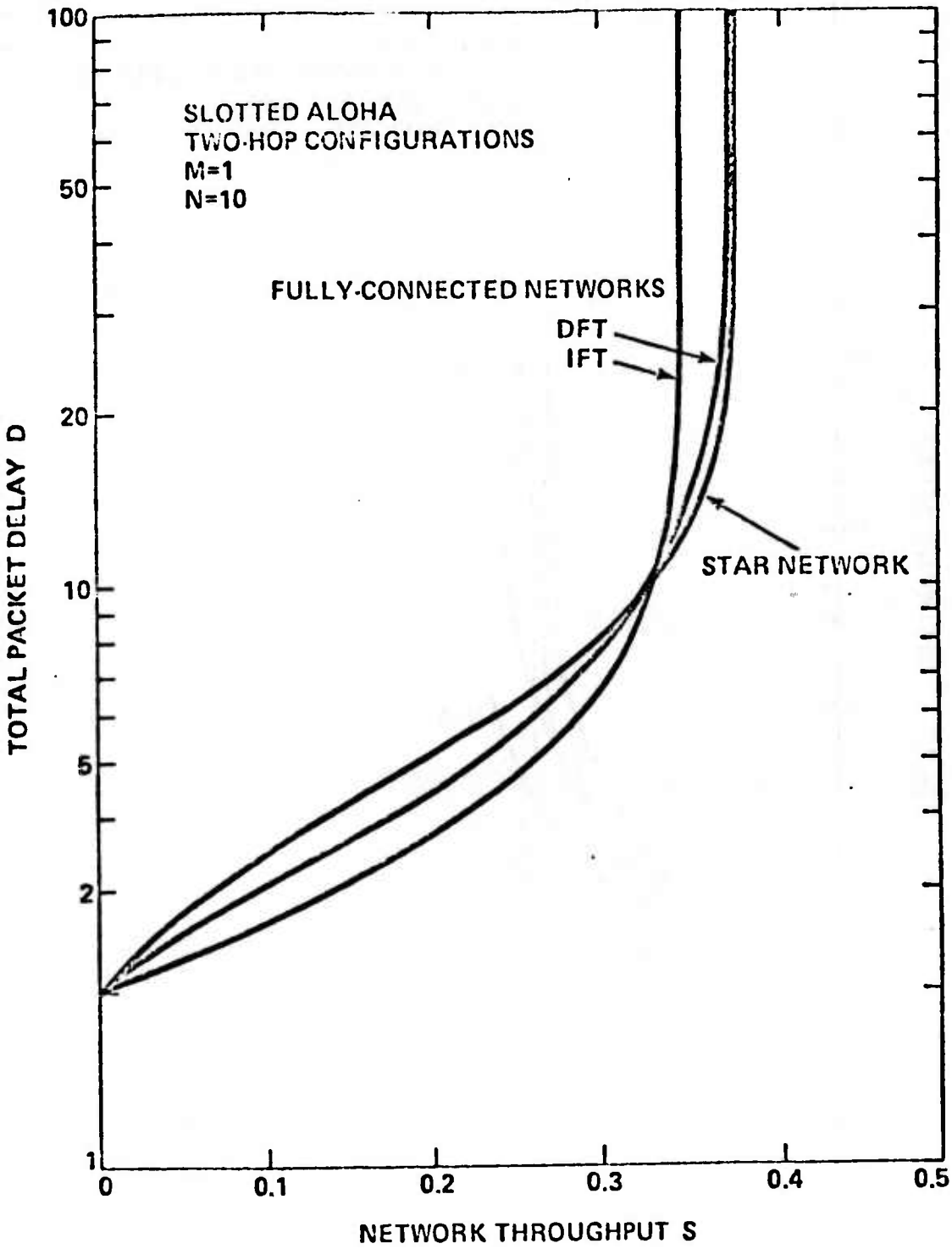

Figure 9 Throughput-Delay Tradeoff in Two-Hop Slotted ALOHA Star and Fully-Connected Networks. (N = 10)

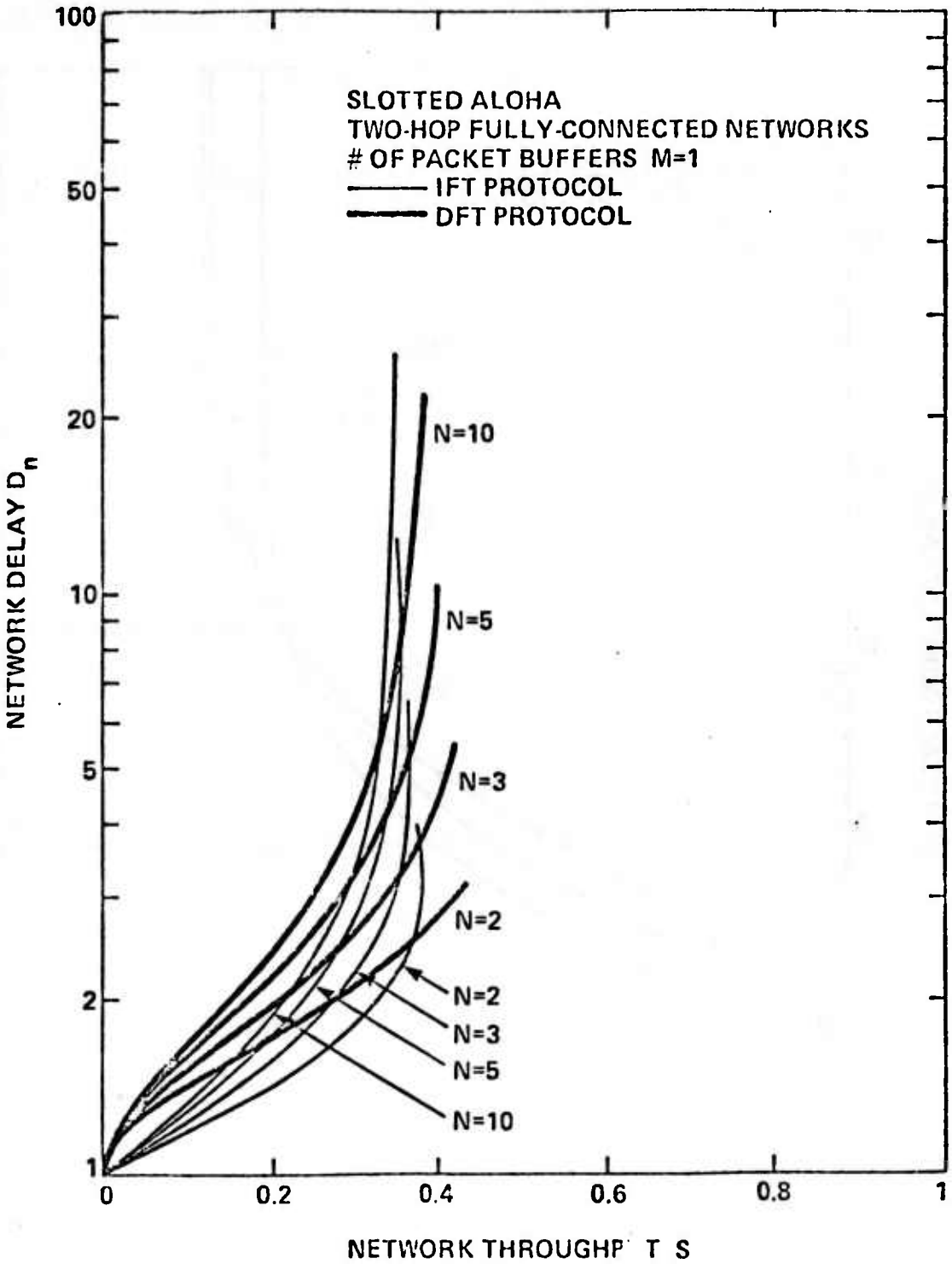

Figure 10 Slotted ALOHA Fully-Connected Configuration: Optimum (Network) Throughput-Delay Curves.

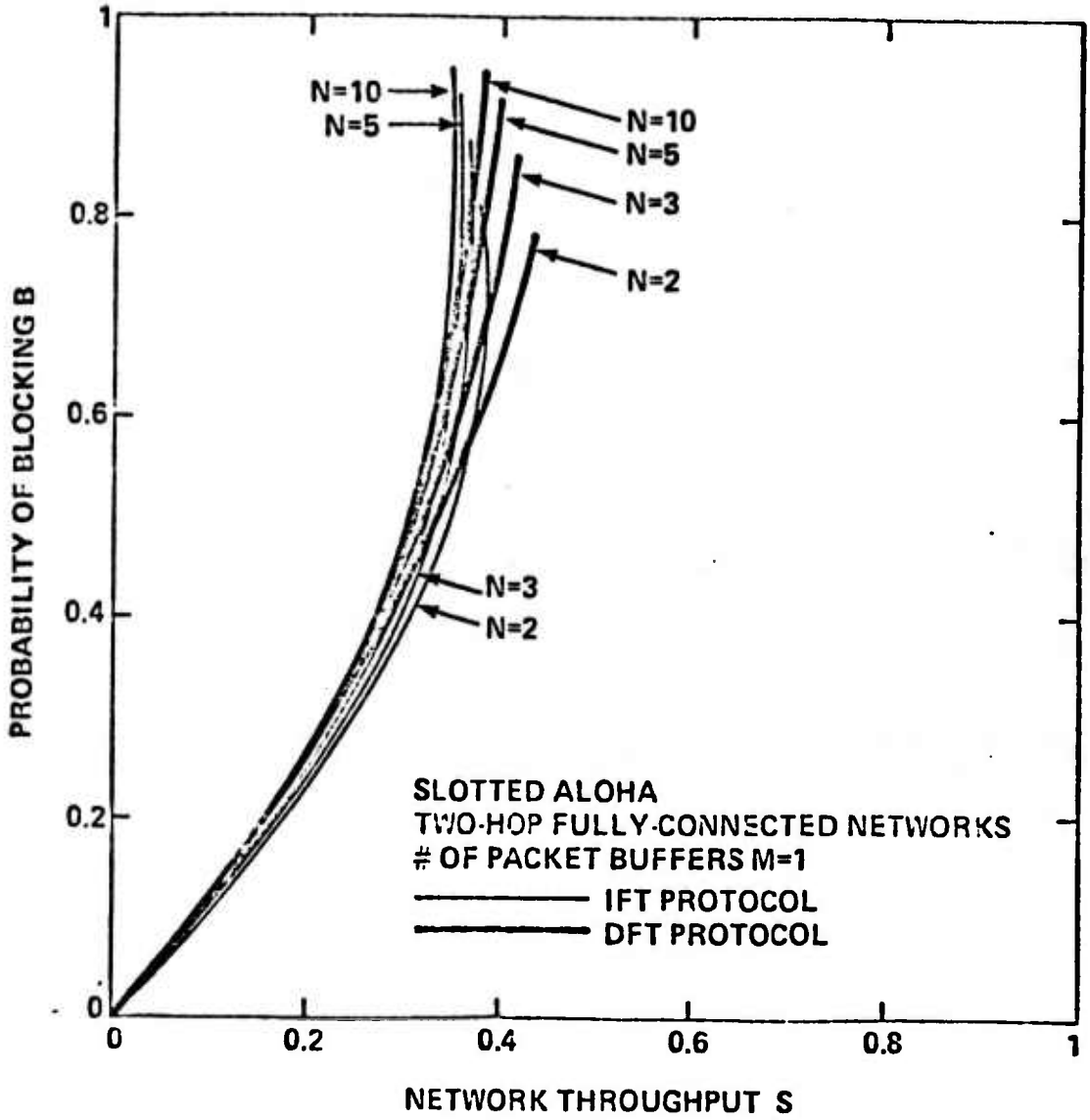

Figure 11 Slotted ALOHA Fully-Connected Configuration: Minimum Blocking Versus S.